Exercise 54

(a) Graph the function

$$f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$$

How many horizontal and vertical asymptotes do you observe? Use the graph to estimate the values of the limits

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5}$$

(b) By calculating values of $f(x)$, give numerical estimates of the limits in part (a).

(c) Calculate the exact values of the limits in part (a). Did you get the same value or different values for these two limits? [In view of your answer to part (a), you might have to check your calculation for the second limit.]

Solution

Determine the horizontal asymptotes by calculating the limits of $f(x)$ as $x \to \pm\infty$. In the second limit, make the substitution, $u = -x$, so that as $x \to -\infty$, $u \to \infty$.

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\sqrt{x^2\left(2+\frac{1}{x^2}\right)}}{3x-5} = \lim_{x\to\infty} \frac{x\sqrt{2+\frac{1}{x^2}}}{3x-5} = \lim_{x\to\infty} \frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} = \frac{\sqrt{2+0}}{3-0} = \frac{\sqrt{2}}{3}$$

$$\begin{aligned}
\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} &= \lim_{u\to\infty} \frac{\sqrt{2(-u)^2+1}}{3(-u)-5} \\
&= \lim_{u\to\infty} \frac{\sqrt{2u^2+1}}{-3u-5} \\
&= \lim_{u\to\infty} \frac{\sqrt{u^2\left(2+\frac{1}{u^2}\right)}}{-3u-5} \\
&= \lim_{u\to\infty} \frac{u\sqrt{2+\frac{1}{u^2}}}{-3u-5} \\
&= \lim_{u\to\infty} \frac{\sqrt{2+\frac{1}{u^2}}}{-3-\frac{5}{u}} \\
&= \frac{\sqrt{2+0}}{-3-0} \\
&= -\frac{\sqrt{2}}{3}
\end{aligned}$$

Therefore, the horizontal asymptotes are $y = \frac{\sqrt{2}}{3}$ and $y = -\frac{\sqrt{2}}{3}$.

To find the vertical asymptotes, set what's in the denominator equal to zero and solve for x.

$$3x - 5 = 0$$

$$3x = 5$$

$$x = \frac{5}{3}$$

Make a table with large positive and negative values of x to see what happens as $x \to \pm\infty$. Note that $\sqrt{2}/3 \approx -0.471405$.

x	$f(x)$
$-10\,000$	-0.471326
$-1\,000$	-0.47062
-100	-0.463688
-10	-0.40507
10	0.567098
100	0.479406
$1\,000$	0.472192
$10\,000$	0.471483

Below is a graph of $f(x)$ versus x with the asymptotes labelled.